Generalized twist-deformed Rindler space-times

Marcin Daszkiewicz

Institute of Theoretical Physics
University of Wrocław pl. Maxa Borna 9, 50-206 Wrocław, Poland
e-mail: marcin@ift.uni.wroc.pl

Abstract

The (linearized) quantum Rindler space-times associated with generalized twist-deformed Minkowski spaces are provided. The corresponding corrections to the Hawking spectra linear in deformation parameters are derived.

1 Introduction

Presently, it is well known that there exist the deep (and extraordinary) relation between horizons of black hole and thermodynamics. Already in early 1970s there was observed by Bekenstein (see [1]), that laws of black hole dynamics (especially the second one) can be given thermodynamical interpretation, if one identifies entropy with the area of black hole horizon and temperature with its "surface gravity". Such an observation has been confirmed by Hawking in his two seminal articles [2], [3], in which, it was predicted that a black hole should radiate with a temperature

$$T_{\text{Black Hole}} = \frac{\hbar g}{2\pi kc}\,, \tag{1}$$

where g denotes the gravitational acceleration at the surface of the black hole, k is Boltzman's constant, and c is the speed of light. Subsequently, it was shown separately by Davies [4] and Unruh [5], that uniformly accelerated observer in vacuum detects a radiation (a thermal field) with the same temperature as $T_{\text{Black Hole}}$

$$T_{\text{Vacuum}} = \frac{\hbar a}{2\pi kc}\,, \tag{2}$$

but with inserted acceleration of the detector a. Formally, such an observer "lives" in so-called Rindler space-time [6], which can be obtained by the following transformation from Minkowski space with coordinates (x_0, x_1, x_2, x_3)[1]

$$x_0 = N(z_1)\sinh(az_0)\,, \tag{3}$$

$$x_1 = N(z_1)\cosh(az_0)\,, \tag{4}$$

$$x_2 = z_2\,, \tag{5}$$

$$x_3 = z_3\,, \tag{6}$$

where N is a positive function of the coordinate. The Minkowski metric $ds^2 = -dx_0^2 + \sum_{i=1}^{3} dx_i^2$ transforms to

$$ds^2 = -aN^2(z_1)dz_0^2 + (N')^2(z_1)dz_1^2 + dz_2^2 + dz_3^2\,. \tag{7}$$

Recently, in the papers [7] and [8], there was proposed the noncommutative counterpart of Rindler space - so-called (linearized) κ-Rindler space and twist-deformed Rindler space-time, respectively. First of them is associated with the well-known κ-deformed Minkowski space [9], [10][2], while the second one with twisted canonical, Lie-algebraic and quadratic quantum Minkowski space-time. Further, following the content of the papers [1]-[5] (see also [11], [12]), there have been found corrections to the Hawking thermal spectrum linear in deformation parameter, which are detected by (noncommutative and uniformly accelerated) κ- as well as twist-deformed Rindler observers.

[1] $c = 1$.

[2] κ denotes the mass-like parameter identified with Planck's mass.

The suggestion to use noncommutative coordinates goes back to Heisenberg and was firstly formalized by Snyder in [13]. Recently, however, the interest in space-time noncommutativity is growing rapidly. Such a situation follows from many phenomenological suggestions, which state that relativistic space-time symmetries should be modified (deformed) at Planck scale, while the classical Poincare invariance still remains valid at larger distances [14]-[17]. Besides, there have been found formal arguments, based mainly on Quantum Gravity [18], [19] and String Theory models [20], [21], indicating that space-time at Planck-length should be noncommutative, i.e. it should have a quantum nature.

At present, in accordance with the Hopf-algebraic classification of all deformations of relativistic and nonrelativistic symmetries [22], [23], one can distinguish three kinds of quantum spaces. First of them corresponds to the well-known canonical type of noncommutativity

$$[\, \hat{x}_\mu, \hat{x}_\nu \,] = i\theta_{\mu\nu} \;, \tag{8}$$

with antisymmetric constant tensor $\theta^{\mu\nu}$. Its relativistic and nonrelativistic Hopf-algebraic counterparts have been proposed in [24] and [25] respectively.
The second kind of mentioned deformations introduces the Lie-algebraic type of space-time noncommutativity

$$[\, \hat{x}_\mu, \hat{x}_\nu \,] = i\theta_{\mu\nu}^{\rho}\hat{x}_\rho \;, \tag{9}$$

with particularly chosen coefficients $\theta_{\mu\nu}^{\rho}$ being constants. The corresponding Poincare quantum groups have been introduced in [26]-[29], while the suitable Galilei algebras - in [30] and [25].
The last kind of quantum space, so-called quadratic type of noncommutativity

$$[\, \hat{x}_\mu, \hat{x}_\nu \,] = i\theta_{\mu\nu}^{\rho\tau}\hat{x}_\rho\hat{x}_\tau \quad ; \quad \theta_{\mu\nu}^{\rho\tau} = \text{const.} \;, \tag{10}$$

has been proposed in [31], [32] and [29] at relativistic and in [33] at nonrelativistic level.

Recently, there was considered the new type of quantum space - so-called generalized quantum space-time

$$[\, \hat{x}_\mu, \hat{x}_\nu \,] = i\theta_{\mu\nu} + i\theta_{\mu\nu}^{\rho}\hat{x}_\rho \;, \tag{11}$$

which combines canonical type with the Lie-algebraic kind of space-time noncommutativity. Its Hopf-algebraic realization has been proposed in [34]-[36] in the case of relativistic symmetry and in [36] for its nonrelativistic counterpart.

In this article, following the scheme proposed in [7] and [8], we provide the noncommutative counterparts of Rindler space-time, associated with generalized twist-deformed Poincare Hopf algebras [36] (see space-time (11)). Further, we investigate the gravito-thermodynamical radiation detected by such generalized twist-deformed Rindler observers in the vacuum, i.e. we find the thermal (Hawking) spectra for twisted space-time (11). Particulary, for parameter $\theta_{\mu\nu}^{\rho}$ approaching zero, we get the thermal spectra for canonical space-time (8) derived in [8].

The paper is organized as follows. In first section we recall the basic facts concerning the generalized twist-deformed Poincare Hopf algebras and the corresponding quantum space-times [36]. The second section is devoted to the generalized Rindler spaces, obtained from their noncommutative Minkowski counterparts. The deformed Hawking radiation

spectra detected by twisted Rindler observers are derived in section three. The final remarks are discussed in the last section.

2 Generalized twist-deformed Minkowski spaces and the corresponding Poincare Hopf structures

In this section, following the paper [36], we recall basic facts related with the generalized twist-deformed relativistic symmetries and corresponding quantum space-times.

In accordance with the general twist procedure [37]-[40], the algebraic sectors of all discussed below Hopf structures remain undeformed ($\eta_{\mu\nu} = (-,+,+,+)$)

$$[M_{\mu\nu}, M_{\rho\sigma}] = i\,(\eta_{\mu\sigma}\,M_{\nu\rho} - \eta_{\nu\sigma}\,M_{\mu\rho} + \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma})\ ,$$
$$[M_{\mu\nu}, P_{\rho}] = i\,(\eta_{\nu\rho}\,P_{\mu} - \eta_{\mu\rho}\,P_{\nu}) \quad , \quad [P_{\mu}, P_{\nu}] = 0\ , \tag{12}$$

while the coproducts and antipodes transform as follows

$$\Delta_0(a) \to \Delta_{\cdot}(a) = \mathcal{F}_{\cdot} \circ\, \Delta_0(a)\, \circ \mathcal{F}_{\cdot}^{-1} \quad , \quad S_{\cdot}(a) = u_{\cdot}\, S_0(a)\, u_{\cdot}^{-1}\ , \tag{13}$$

where $\Delta_0(a) = a\otimes 1 + 1\otimes a$, $S_0(a) = -a$ and $u_{\cdot} = \sum f_{(1)}S_0(f_{(2)})$ (we use Sweedler's notation $\mathcal{F}_{\cdot} = \sum f_{(1)} \otimes f_{(2)}$). Present in the above formula the twist element $\mathcal{F}_{\cdot} \in \mathcal{U}_{\cdot}(\mathcal{P}) \otimes \mathcal{U}_{\cdot}(\mathcal{P})$ satisfies the classical cocycle condition

$$\mathcal{F}_{\cdot 12} \cdot (\Delta_0 \otimes 1)\ \mathcal{F}_{\cdot} = \mathcal{F}_{\cdot 23} \cdot (1 \otimes \Delta_0)\ \mathcal{F}_{\cdot}\ , \tag{14}$$

and the normalization condition

$$(\epsilon \otimes 1)\ \mathcal{F}_{\cdot} = (1 \otimes \epsilon)\ \mathcal{F}_{\cdot} = 1\ , \tag{15}$$

with $\mathcal{F}_{\cdot 12} = \mathcal{F}_{\cdot} \otimes 1$ and $\mathcal{F}_{\cdot 23} = 1 \otimes \mathcal{F}_{\cdot}$.

The corresponding to the above Hopf structure space-time is defined as quantum representation space (Hopf module) for quantum Poincare algebra, with action of the deformed symmetry generators satisfying suitably deformed Leibnitz rules [41], [42], [24]. The action of Poincare algebra on its Hopf module of functions depending on space-time coordinates x_μ is given by

$$P_{\mu} \rhd f(x) = i\partial_{\mu} f(x) \quad , \quad M_{\mu\nu} \rhd f(x) = i\,(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})\, f(x)\ , \tag{16}$$

while the $\star_{\cdot}$-multiplication of arbitrary two functions is defined as follows

$$f(x) \star_{\cdot} g(x) := \omega \circ \left(\mathcal{F}_{\cdot}^{-1} \rhd f(x) \otimes g(x)\right)\ . \tag{17}$$

In the above formula $\mathcal{F}_{\cdot}$ denotes twist factor in the differential representation (16) and $\omega \circ (a \otimes b) = a \cdot b$.

In the article [36], there have been considered three (all possible) types of Abelian and generalized twist factors ($a \wedge b = a \otimes b - b \otimes a$)[3]:

$$i) \quad \mathcal{F}_{\theta_{kl},\kappa} = \exp\ i\left[\frac{1}{2\kappa}P_k \wedge M_{i0} + \theta_{kl}P_k \wedge P_l\right]\ , \tag{18}$$

$$ii) \quad \mathcal{F}_{\theta_{0i},\hat{\kappa}} = \exp\ i\left[\frac{1}{2\hat{\kappa}}P_0 \wedge M_{kl} + \theta_{0i}P_0 \wedge P_i\right]\ , \tag{19}$$

and

$$iii) \quad \mathcal{F}_{\theta_{0i},\bar{\kappa}} = \exp\ i\left[\frac{1}{2\bar{\kappa}}P_i \wedge M_{kl} + \theta_{0i}P_0 \wedge P_i\right]\ , \tag{20}$$

leading to the following generalized quantum space-times (see (11)):

$$i) \qquad [\,x_0, x_a\,]_{\star_{\theta_{kl},\kappa}} = \frac{i}{\kappa}x_i\delta_{ak}\ , \tag{21}$$
$$[\,x_a, x_b\,]_{\star_{\theta_{kl},\kappa}} = 2i\theta_{kl}(\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk}) + \frac{i}{\kappa}x_0(\delta_{ia}\delta_{kb} - \delta_{ka}\delta_{ib})\ ,$$

$$ii) \qquad [\,x_0, x_a\,]_{\star_{\theta_{0i},\hat{\kappa}}} = \frac{i}{\hat{\kappa}}(\delta_{la}x_k - \delta_{ka}x_l) + 2i\theta_{0i}\delta_{ia}\ ,$$
$$[\,x_a, x_b\,]_{\star_{\theta_{0i},\hat{\kappa}}} = 0\ , \tag{22}$$

and

$$iii) \qquad [\,x_0, x_a\,]_{\star_{\theta_{0i},\bar{\kappa}}} = 2i\theta_{0i}\delta_{ia}\ ,$$
$$[\,x_a, x_b\,]_{\star_{\theta_{0i},\bar{\kappa}}} = \frac{i}{\bar{\kappa}}\delta_{ib}(\delta_{ka}x_l - \delta_{la}x_k) + \frac{i}{\bar{\kappa}}\delta_{ia}(\delta_{lb}x_k - \delta_{kb}x_l)\ , \tag{23}$$

respectively, with star product given by the formula (17).

The corresponding Poincare Hopf structures have been provided in [36] as well. However, due to their complicated form, in this article, we recall as an example only one Poincare Hopf algebra, associated with first twist factor (18). In accordance with mentioned above twist procedure its algebraic sector remains classical (see formula (12)), while the coproducts take the form (see formula (13))

$$\begin{aligned}\Delta_{\theta_{kl},\kappa}(P_\mu) &= \Delta_0(P_\mu) + \sinh(\frac{1}{2\kappa}P_k) \wedge (\eta_{i\mu}P_0 - \eta_{0\mu}P_i) \\ &+ (\cosh(\frac{1}{2\kappa}P_k) - 1) \perp (\eta_{i\mu}P_i - \eta_{0\mu}P_0)\ ,\end{aligned} \tag{24}$$

[3] Indecies k, l are fixed and different than i.

$$
\begin{aligned}
\Delta_{\theta_{kl},\kappa}(M_{\mu\nu}) &= \Delta_0(M_{\mu\nu}) + \frac{1}{2\kappa}M_{i0}\wedge(\eta_{\mu k}P_\nu - \eta_{\nu k}P_\mu) \\
&+ i\,[M_{\mu\nu}, M_{i0}]\wedge\sinh(\frac{1}{2\kappa}P_k) \\
&- [[M_{\mu\nu}, M_{i0}]\,, M_{i0}]\perp(\cosh(\frac{1}{2\kappa}P_k)-1) \\
&+ \frac{1}{2\kappa}M_{i0}\sinh(\frac{1}{2\kappa}P_k)\perp(\psi_k P_i - \chi_k P_0) \\
&- \frac{1}{2\kappa}(\psi_k P_0 - \chi_k P_i)\wedge M_{i0}(\cosh(\frac{1}{2\kappa}P_k)-1)
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
&- \theta^{kl}[(\eta_{k\mu}P_\nu - \eta_{k\nu}P_\mu)\otimes P_l + P_k\otimes(\eta_{l\mu}P_\nu - \eta_{l\nu}P_\mu)] \\
&+ \theta_{kl}[(\eta_{l\mu}P_\nu - \eta_{l\nu}P_\mu)\otimes P_k + P_l\otimes(\eta_{k\mu}P_\nu - \eta_{k\nu}P_\mu)] \\
&+ \theta_{kl}\,[[M_{\mu\nu}, M_{i0}]\,, P_k]\perp\sinh(\frac{1}{2\kappa}P_k)P_l \\
&- \theta_{kl}\,[[M_{\mu\nu}, M_{i0}]\,, P_l]\perp\sinh(\frac{1}{2\kappa}P_k)P_k \\
&+ i\theta_{kl}\,[[[M_{\mu\nu}, M_{i0}]\,, M_{i0}]\,, P_k]\wedge(\cosh(\frac{1}{2\kappa}P_k)-1)P_l \\
&- i\theta_{kl}\,[[[M_{\mu\nu}, M_{i0}]\,, M_{i0}]\,, P_l]\wedge(\cosh(\frac{1}{2\kappa}P_k)-1)P_k\;,
\end{aligned}
$$

with $a\perp b = a\otimes b + b\otimes a$, $\psi_\gamma = \eta_{j\gamma}\eta_{li} - \eta_{i\gamma}\eta_{lj}$ and $\chi_\gamma = \eta_{j\gamma}\eta_{ki} - \eta_{i\gamma}\eta_{kj}$. The two remaining Hopf structures corresponding to the twist factors (19) and (20) look similar to the coproducts (24) and (25).

Of course, if parameter $\theta^{\mu\nu}$ goes to zero and parameters κ, $\hat\kappa$ and $\bar\kappa$ approach infinity, the above space-times and corresponding Hopf structures become undeformed. Besides, for fixed (different than zero) parameters θ^{kl} and θ^{0i}, and parameters κ, $\hat\kappa$ and $\bar\kappa$ approaching infinity, we get twisted canonical Minkowski space provided in [24]. Moreover, for parameters θ^{kl} and θ^{0i} running to zero, and fixed parameters κ, $\hat\kappa$ and $\bar\kappa$, we recover the Lie-algebraically deformed relativistic spaces introduced in [28] and [29].

3 Generalized twist-deformed Rindler space-times

Let us now find the twisted Rindler spaces associated with the generalized Minkowski space-times provided in pervious section. In this aim we proceed with the algorithm proposed in [7] and [8] for κ- and twist-deformed Minkowski space-times, respectively.

We define such (Rindler) space-times as the quantum spaces with noncommutativity given by the proper $*$-multiplications. This new $*$-multiplications are defined by the new $\mathcal{Z}$-factors, which can be get from relativistic twist factors (18)-(20) as follows:

i) Firstly, we take the standard transformation rules from commutative Minkowski space (described by x_μ variables) to the accelerated and commutative as well (Rindler)

space-time (z_μ) [6]

$$
\begin{aligned}
x_0 &= z_1 \sinh(a z_0) \ , && (26)\\
x_1 &= z_1 \cosh(a z_0) \ , && (27)\\
x_2 &= z_2 \ , && (28)\\
x_3 &= z_3 \ , && (29)
\end{aligned}
$$

where a denotes the acceleration parameter, i.e. we have chosen function $N(z_1) = z_1$ in formulas (3)-(6).

ii) Further, we rewrite the Minkowski twist factors (18)-(20) (depending on commutative x_μ variables and defining the $\star$-multiplication (17)) in terms of z_μ variables.

In such a way, we get three following $\mathcal{Z}$-factors and the corresponding Rindler quantum spaces:

i) Rindler space-time associated with twisted relativistic space i) (see formula (21)).

In such a case, due to the transformation rules (26)-(29)[4], the proper $*_{\theta_{kl},\kappa}$-product takes the form

$$
f(z) *_{\theta_{kl},\kappa} g(z) := \omega \circ \left(\mathcal{Z}^{-1}_{\theta_{kl},\kappa} \rhd f(z) \otimes g(z) \right) \ , \qquad (30)
$$

where

$$
\begin{aligned}
\mathcal{Z}^{-1}_{\theta_{kl},\kappa} &= \exp -i\,(\delta_{k1}\theta_{1l}\ f_1(z_0,z_1,\partial_{z_0},\partial_{z_1}) \wedge \partial_{z_l} + \delta_{l1}\theta_{k1}\ \partial_{z_k} \wedge f_1(z_0,z_1,\partial_{z_0},\partial_{z_1}) +\\
&+ \ \delta_{k2}\delta_{l3}\theta_{23}\ \partial_{z_2} \wedge \partial_{z_3} + \delta_{k3}\delta_{l2}\theta_{32}\ \partial_{z_3} \wedge \partial_{z_2} +\\
&+ \ \frac{1}{2\kappa}\delta_{k1} f_1(z_0,z_1,\partial_{z_0},\partial_{z_1}) \wedge (z_i f_0(z_0,z_1,\partial_{z_0},\partial_{z_1}) - g_0(z_0,z_1)\partial_{z_i}) +\\
&+ \ \frac{1}{2\kappa}\delta_{i1}\ \partial_{z_k} \wedge (g_1(z_0,z_1) f_0(z_0,z_1,\partial_{z_0},\partial_{z_1}) - g_0(z_0,z_1) f_1(z_0,z_1,\partial_{z_0},\partial_{z_1})) \qquad (31)\\
&+ \ \frac{1}{2\kappa}\delta_{k2}\delta_{i3}\ \partial_{z_2} \wedge (z_3 f_0(z_0,z_1,\partial_{z_0},\partial_{z_1}) - g_0(z_0,z_1)\partial_{z_3}) +\\
&+ \ \frac{1}{2\kappa}\delta_{k3}\delta_{i2}\ \partial_{z_3} \wedge (z_2 f_0(z_0,z_1,\partial_{z_0},\partial_{z_1}) - g_0(z_0,z_1)\partial_{z_2}) \Big) =\\
&= \exp\left(\mathcal{A}_{\theta_{kl},\kappa}(z,\partial_z) \wedge \mathcal{B}_{\theta_{kl},\kappa}(z,\partial_z)\right) = \exp \mathcal{O}_{\theta_{kl},\kappa}(z,\partial_z) \ ,
\end{aligned}
$$

and

$$
\begin{aligned}
f_0(z_0,z_1,\partial_{z_0},\partial_{z_1}) &= -\sinh(a z_0) i\partial_{z_1} + (\cosh(a z_0)/a z_1) i\partial_{z_0} \ , && (32)\\
f_1(z_0,z_1,\partial_{z_0},\partial_{z_1}) &= \cosh(a z_0) i\partial_{z_1} - (\sinh(a z_0)/a z_1) i\partial_{z_0} \ , && (33)\\
g_0(z_0,z_1) &= z_1 \sinh(a z_0) \quad , \quad g_1(z_0,z_1) = z_1 \cosh(a z_0) \ . && (34)
\end{aligned}
$$

[4]One can find, that $\partial_{x_0} = (-\sinh(a z_0)\partial_{z_1} + (\cosh(a z_0)/a z_1)\partial_{z_0})$ and $\partial_{x_1} = (\cosh(a z_0)\partial_{z_1} - (\sinh(a z_0) a z_1)\partial_{z_0})$.

However, to simplify, we consider the following differential operator

$$(\mathcal{Z}_{\theta_{kl},\kappa}^{\mathrm{Linear}})^{-1} = 1 + \mathcal{O}_{\theta_{kl},\kappa}(z,\partial_z)\ , \tag{35}$$

which contains only the terms linear in deformation parameter θ^{kl} and κ[5]. Hence, the linearized $\hat{*}$-Rindler multiplication is given by the formula (30), but with differential operator (35) instead the complete factor $\mathcal{Z}^{-1}_{\theta_{kl},\kappa}$. Consequently, for $f(z) = z_\mu$ and $g(z) = z_\nu$, we get

$$[\ z_\mu, z_\nu\ ]_{\hat{*}_{\theta_{kl}},\kappa} = [(\mathcal{A}_{\theta_{kl},\kappa}(z,\partial_z)z_\mu)(\mathcal{B}_{\theta_{kl},\kappa}(z,\partial_z)z_\nu) - (\mathcal{B}_{\theta_{kl},\kappa}(z,\partial_z)z_\mu)(\mathcal{A}_{\theta_{kl},\kappa}(z,\partial_z)z_\nu)] \tag{36}$$

with $f_2(z,\partial_z) = i\partial_{z_2},\ f_3(z,\partial_z) = i\partial_{z_3}$. The above commutation relations define the generalized twist-deformed Rindler space-time associated with generalized Minkowski space (21).

ii) Rindler space-time associated with generalized twist-deformed Minkowski space *ii*) (see (22)).

Here, due to the rules (26)-(29), the $*_{\theta_{0i},\hat{\kappa}}$-multiplication look as follows

$$f(z) *_{\theta_{0i},\hat{\kappa}} g(z) = \omega \circ \left(\mathcal{Z}^{-1}_{\theta_{0i},\hat{\kappa}} \rhd f(z)\otimes g(z)\right)\ , \tag{37}$$

where

$$\begin{aligned}
\mathcal{Z}^{-1}_{\theta_{0i},\hat{\kappa}} &= \exp -i\,(\delta_{i1}\theta_{01}\ f_0(z_0,z_1,\partial_{z_0},\partial_{z_1}) \wedge f_1(z_0,z_1,\partial_{z_0},\partial_{z_1}) + \\
&+ \delta_{i2}\theta_{02}\ f_0(z_0,z_1,\partial_{z_0},\partial_{z_1}) \wedge \partial_{z_2} + \delta_{i3}\theta_{03}\ f_0(z_0,z_1,\partial_{z_0},\partial_{z_1}) \wedge \partial_{z_3} + \\
&+ \frac{1}{2\hat{\kappa}}\delta_{k2}\delta_{l3}\ f_0(z_0,z_1,\partial_{z_0},\partial_{z_1}) \wedge (z_2\partial_{z_3} - z_3\partial_{z_2}) + \\
&+ \frac{1}{2\hat{\kappa}}\delta_{k3}\delta_{l2}\ f_0(z_0,z_1,\partial_{z_0},\partial_{z_1}) \wedge (z_3\partial_{z_2} - z_2\partial_{z_3}) + \\
&+ \frac{1}{2\hat{\kappa}}\delta_{k1}\ f_0(z_0,z_1,\partial_{z_0},\partial_{z_1}) \wedge (g_1(z_0,z_1)\partial_{z_l} - z_l f_1(z_0,z_1,\partial_{z_0},\partial_{z_1})) + \\
&+ \frac{1}{2\hat{\kappa}}\delta_{l1}\ f_0(z_0,z_1,\partial_{z_0},\partial_{z_1}) \wedge (z_k f_1(z_0,z_1,\partial_{z_0},\partial_{z_1}) - g_1(z_0,z_1)\partial_{z_k})\Big) = \\
&= \exp\left(\mathcal{C}_{\theta_{0i},\hat{\kappa}}(z,\partial_z) \wedge \mathcal{D}_{\theta_{0i},\hat{\kappa}}(z,\partial_z)\right) = \exp \mathcal{O}_{\theta_{0i},\hat{\kappa}}(z,\partial_z)\ ,
\end{aligned} \tag{38}$$

Consequently, the corresponding (linearized) Rindler space-time takes the form

$$[\ z_\mu, z_\nu\ ]_{\hat{*}_{\theta_{0i},\hat{\kappa}}} = [(\mathcal{C}_{\theta_{0i},\hat{\kappa}}(z,\partial_z)z_\mu)(\mathcal{D}_{\theta_{0i},\hat{\kappa}}(z,\partial_z)z_\nu) - (\mathcal{D}_{\theta_{0i},\hat{\kappa}}(z,\partial_z)z_\mu)(\mathcal{C}_{\theta_{0i},\hat{\kappa}}(z,\partial_z)z_\nu)] \tag{39}$$

where we use the linearized approximation to $\mathcal{Z}^{-1}_{\theta_{0i},\hat{\kappa}}$ (see (35)).

iii) Rindler space-time associated with relativistic twist-deformed space *iii*) (see formula (23)).

[5] We look only for the corrections to Hawking radiation linear in deformation parameter.

In such a case, the $*_{\theta_{0i},\bar{\kappa}}$-multiplication takes the form

$$f(z) *_{\theta_{0i},\bar{\kappa}} g(z) = \omega \circ \left(\mathcal{Z}_{\theta_{0i},\bar{\kappa}}^{-1} \triangleright f(z) \otimes g(z) \right) , \tag{40}$$

with factor

$$\begin{aligned}
\mathcal{Z}_{\theta_{0i},\bar{\kappa}}^{-1} &= \exp -i \,(\delta_{i1}\theta_{01}\; f_0(z_0, z_1, \partial_{z_0}, \partial_{z_1}) \wedge f_1(z_0, z_1, \partial_{z_0}, \partial_{z_1}) + \\
&+ \delta_{i2}\theta_{02}\; f_0(z_0, z_1, \partial_{z_0}, \partial_{z_1}) \wedge \partial_{z_2} + \delta_{i3}\theta_{03}\; f_0(z_0, z_1, \partial_{z_0}, \partial_{z_1}) \wedge \partial_{z_3} + \\
&+ \frac{1}{2\bar{\kappa}}\delta_{i1}\; f_1(z_0, z_1, \partial_{z_0}, \partial_{z_1}) \wedge (z_k \partial_{z_l} - z_l \partial_{z_k}) + \\
&+ \frac{1}{2\bar{\kappa}}\delta_{k1}\; \partial_{z_i} \wedge (g_1(z_0, z_1)\partial_{z_l} - z_l f_1(z_0, z_1, \partial_{z_0}, \partial_{z_1})) + \\
&+ \frac{1}{2\bar{\kappa}}\delta_{l1}\; \partial_{z_i} \wedge (z_k f_1(z_0, z_1, \partial_{z_0}, \partial_{z_1}) - g_1(z_0, z_1)\partial_{z_k}) \Big) = \\
&= \exp\left(\mathcal{E}_{\theta_{0i},\bar{\kappa}}(z, \partial_z) \wedge \mathcal{G}_{\theta_{0i},\bar{\kappa}}(z, \partial_z)\right) = \exp \mathcal{O}_{\theta_{0i},\bar{\kappa}}(z, \partial_z) \; .
\end{aligned} \tag{41}$$

Then, the (linearized) Rindler space looks as follows

$$[\, z_\mu, z_\nu \,]_{\hat{*}_{\theta_{0i},\bar{\kappa}}} = [(\mathcal{E}_{\theta_{0i},\bar{\kappa}}(z, \partial_z) z_\mu)(\mathcal{G}_{\theta_{0i},\bar{\kappa}}(z, \partial_z) z_\nu) - (\mathcal{G}_{\theta_{0i},\bar{\kappa}}(z, \partial_z) z_\mu)(\mathcal{E}_{\theta_{0i},\bar{\kappa}}(z, \partial_z) z_\nu)] \tag{42}$$

with $\hat{*}_{\theta_{0i},\bar{\kappa}}$-multiplication defined by the linear approximation to (41) (see (35)).

Obviously, for both deformation parameters θ^{kl} and θ^{0i} approaching zero, and all parameters κ, $\hat{\kappa}$ and $\bar{\kappa}$ running to infinity, the above generalized Rindler space-times become classical. It should be also noted, that for fixed (different than zero) parameters θ^{kl} and θ^{0i}, and parameters κ, $\hat{\kappa}$ and $\bar{\kappa}$ approaching infinity, we get twisted canonical Rindler space provided in [8]. On the other side, for parameters θ^{kl} and θ^{0i} running to zero, and fixed parameters κ, $\hat{\kappa}$ and $\bar{\kappa}$, we recover the Lie-algebraically deformed Rindler spaces introduced in [8] as well.

4 Hawking thermal spectra for generalized twist-deformed Rindler space-times

In this section we find the corrections to the gravito-thermodynamical process, which occur in generalized twist-deformed space-times.

As it was mentioned in Introduction, such effects as Hawking radiation [2], can be observed in vacuum by uniformly accelerated observer [4], [5]. First of all, following [7] and [8], we recall the calculations performed for gravito-thermodynamical process in commutative relativistic space-time [11], [12]. Firstly, we consider the on-shell plane wave corresponding to the massless mode with positive frequency $\hat{\omega}$ moving in $x_1 = x$ direction of Minkowski space ($x_0 = t$)

$$\phi(x, t) = \exp\left(\hat{\omega} x - \hat{\omega} t\right) \; . \tag{43}$$

In terms of Rindler variables this plane wave takes the form ($z_0 = \tau, z_1 = z$)

$$\phi(x(z,\tau), t(z,\tau)) \equiv \phi(z,\tau) = \exp\left(i\hat{\omega}z\mathrm{e}^{-a\tau}\right) \ , \tag{44}$$

i.e. it becomes nonmonochromatic and instead has the frequency spectrum $f(\omega)$, given by Fourier transform

$$\phi(z,\tau) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} f(\omega)\mathrm{e}^{-i\omega\tau} \ . \tag{45}$$

The corresponding power spectrum is given by $P(\omega) = |f(\omega)|^2$ and the function $f(\omega)$ can be obtained by inverse Fourier transform

$$f(\omega) = \int_{-\infty}^{+\infty} d\tau\, \mathrm{e}^{i\hat{\omega}z\mathrm{e}^{-a\tau}}\mathrm{e}^{i\omega\tau} = \left(-\frac{1}{a}\right)(\hat{\omega}z)^{i\omega/a}\,\Gamma\left(-\frac{i\omega}{a}\right)\mathrm{e}^{\pi\omega/2a} \ , \tag{46}$$

where $\Gamma(x)$ denotes the gamma function [43]. Then, since

$$\left|\Gamma\left(\frac{i\omega}{a}\right)\right|^2 = \frac{\pi}{(\omega/a)\mathrm{sinh}(\pi\omega/a)} \ , \tag{47}$$

we get the following power spectrum at negative frequency

$$\omega P(-\omega) = \omega|f(-\omega)|^2 = \frac{2\pi/a}{\mathrm{e}^{2\pi\omega/a} - 1} \ , \tag{48}$$

which corresponds to the Planck factor $\left(\mathrm{e}^{\hbar\omega/kT} - 1\right)$ associated with temperature $T = \hbar a/2\pi kc$ (the temperature of radiation seen by Rindler observer (see formula (2))).

Let us now turn to the case of generalized twist-deformed space-times provided in pervious section. In order to find the power spectra for such deformed Rindler spaces, we start with the (fundamental) formula (44) for scalar field, equipped with the twisted (linearized) $\hat{*}$-multiplications

$$\phi^{\mathrm{Twisted}}_{\cdot,\cdot}(z,\tau) = \exp\left(i\hat{\omega}z\hat{*}_{\cdot,\cdot}\mathrm{e}^{-a\tau}\right) \ . \tag{49}$$

Then

$$f^{\mathrm{Twisted}}_{\cdot,\cdot}(\omega) = \int_{-\infty}^{+\infty} d\tau\, \mathrm{e}^{i\hat{\omega}z\hat{*}_{\cdot,\cdot}\mathrm{e}^{-a\tau}}\,\hat{*}_{\cdot,\cdot}\mathrm{e}^{i\omega\tau} \ , \tag{50}$$

and, in accordance with the pervious considerations, we get[6]

$$\begin{aligned} f^{\mathrm{Twisted}}_{\cdot,\cdot}(\omega) &= f(\omega) + \int_{-\infty}^{+\infty} d\tau\, \omega\circ\left(\mathcal{O}_{\cdot,\cdot}(\tau,z,\partial_\tau,\partial_z)\rhd \mathrm{e}^{i\hat{\omega}z\mathrm{e}^{-a\tau}}\otimes\mathrm{e}^{i\omega\tau}\right) + \\ &+ \int_{-\infty}^{+\infty} d\tau\, \mathrm{e}^{i\hat{\omega}z\mathrm{e}^{-a\tau}}\mathrm{e}^{i\omega\tau}\omega\circ\left(\mathcal{O}_{\cdot,\cdot}(\tau,z,\partial_\tau,\partial_z)\rhd i\hat{\omega}z\otimes\mathrm{e}^{-a\tau}\right) \ ; \end{aligned} \tag{51}$$

[6]We only take under consideration the terms linear in deformation parameters θ^{kl}, θ^{0i}, κ, $\hat{\kappa}$ and $\bar{\kappa}$.

the corresponding (twisted) power spectrum is defined as

$$\omega P^{\rm Twisted}_{\cdot,\cdot}(-\omega) \;\;=\;\; \omega\left|f^{\rm Twisted}_{\cdot,\cdot}(-\omega)\right|^2 \;. \tag{52}$$

Consequently, due to the form of linearized Rindler factors $\mathcal{Z}^{\rm Linear}_{\cdot,\cdot}$, we have:

i) The thermal spectrum for generalized Rindler space *i*).

In such a case the operator $\mathcal{O}_{\cdot,\cdot}(\tau,z,\partial_\tau,\partial_z) = \mathcal{O}_{\theta_{kl},\kappa}(\tau,z,\partial_\tau,\partial_z)$ is given by the formula (31) and

$$\begin{aligned} f^{\rm Twisted}_{\theta_{kl},\kappa}(\omega) \;\;&=\;\; f(\omega) + \frac{i\delta_{k1}\; z_i\;\omega\;\hat{\omega}}{2\kappa a z}\int_{-\infty}^{+\infty} d\tau\, {\rm e}^{i\hat{\omega}z{\rm e}^{-a\tau}}{\rm e}^{i\omega\tau}{\rm e}^{-a\tau} + \\ &\qquad - \frac{\delta_{k1}\; z_i\;\hat{\omega}}{2\kappa z}\int_{-\infty}^{+\infty} d\tau\, {\rm e}^{i\hat{\omega}z{\rm e}^{-a\tau}}{\rm e}^{i\omega\tau}{\rm e}^{-a\tau} \;. \end{aligned} \tag{53}$$

The above integral can be evaluated with help of standard identity for Gamma function

$$\Gamma(y+1) = y\Gamma(y)\;; \tag{54}$$

one gets

$$f^{\rm Twisted}_{\theta_{kl},\kappa}(\omega) = \left(-\frac{1}{a}\right)(\hat{\omega}z)^{i\omega/a}\,\Gamma\left(-\frac{i\omega}{a}\right){\rm e}^{\pi\omega/2a}\left(1+\frac{\delta_{k1}\; z_i\;\omega}{2\kappa a z^2}\left[\frac{i\omega}{a}-1\right]\right) \;. \tag{55}$$

Then, one can check that the thermal spectrum takes the form

$$\omega P^{\rm Twisted}_{\theta_{kl},\kappa}(-\omega) = \frac{1}{T}\frac{1}{{\rm e}^{\omega/T}-1}\left(1-\frac{\delta_{k1}\; z_i\;\omega}{2\kappa\pi T z^2}\right) + \mathcal{O}(\kappa^2)\;, \tag{56}$$

with Hawking temperature $T = a/2\pi$ associated with the acceleration of generalized twist-deformed Rindler observer *i*).

ii) The Hawking radiation spectrum associated with twist deformation *ii*).

Then, $\mathcal{O}_{\cdot,\cdot}(\tau,z,\partial_\tau,\partial_z) = \mathcal{O}_{\theta_{0i},\hat{\kappa}}(\tau,z,\partial_\tau,\partial_z)$ (see formula (38)) and the function $f^{\rm Twisted}_{\cdot,\cdot}(\omega)$ looks as follows

$$\begin{aligned} f^{\rm Twisted}_{\theta_{0i},\hat{\kappa}}(\omega) \;\;&=\;\; f(\omega) + i\left(\theta_{01}+\frac{1}{2\hat{\kappa}}(\delta_{l1}z_k-\delta_{k1}z_l)\right)\frac{\omega\hat{\omega}}{az}\int_{-\infty}^{+\infty} d\tau\, {\rm e}^{i\hat{\omega}z{\rm e}^{-a\tau}}{\rm e}^{i\omega\tau}{\rm e}^{-a\tau} + \\ &\quad - \;\; \left(\theta_{01}+\frac{1}{2\hat{\kappa}}(\delta_{l1}z_k-\delta_{k1}z_l)\right)\frac{\hat{\omega}}{z}\int_{-\infty}^{+\infty} d\tau\, {\rm e}^{i\hat{\omega}z{\rm e}^{-a\tau}}{\rm e}^{i\omega\tau}{\rm e}^{-a\tau} \;. \end{aligned} \tag{57}$$

One can perform the above integral with respect time τ

$$\begin{aligned} f^{\rm Twisted}_{\theta_{0i},\hat{\kappa}}(\omega) \;\;&=\;\; \left(-\frac{1}{a}\right)(\hat{\omega}z)^{i\omega/a}\,\Gamma\left(-\frac{i\omega}{a}\right){\rm e}^{\pi\omega/2a}\,\cdot \\ &\qquad \cdot\left(1+\left(\theta_{01}+\frac{1}{2\hat{\kappa}}(\delta_{l1}z_k-\delta_{k1}z_l)\right)\frac{\omega}{az^2}\left[\frac{i\omega}{a}-1\right]\right) \;, \end{aligned} \tag{58}$$

and, consequently, the power spectrum takes the form

$$\begin{aligned}\omega P^{\text{Twisted}}_{\theta_{0i},\hat\kappa}(-\omega) &= \frac{1}{T}\frac{1}{\mathrm{e}^{\omega/T}-1}\left(1-\left(\theta_{01}+\frac{1}{2\hat\kappa}(\delta_{l1}z_k-\delta_{k1}z_l)\right)\frac{\omega}{\pi T z^2}\right)+\\ &+ \mathcal{O}(\theta_{01}^2,\hat\kappa^2)\ ,\end{aligned} \tag{59}$$

with Hawking temperature $T = a/2\pi$.

iii) The thermal spectrum corresponding to the generalized deformation of Rindler space *iii*).

In the last case $\mathcal{O}_{\cdot,\cdot}(\tau,z,\partial_\tau,\partial_z) = \mathcal{O}_{\theta_{0i},\bar\kappa}(\tau,z,\partial_\tau,\partial_z)$ (see formula (41)) and, we have

$$\begin{aligned}f^{\text{Twisted}}_{\theta_{0i},\bar\kappa}(\omega) &= f(\omega)+\frac{i\delta_{i1}\ \theta_{01}\ \omega\ \hat\omega}{az}\int_{-\infty}^{+\infty}d\tau\,\mathrm{e}^{i\hat\omega z\mathrm{e}^{-a\tau}}\mathrm{e}^{i\omega\tau}\mathrm{e}^{-a\tau}+\\ &\qquad -\frac{\delta_{i1}\ \theta_{01}\ \hat\omega}{z}\int_{-\infty}^{+\infty}d\tau\,\mathrm{e}^{i\hat\omega z\mathrm{e}^{-a\tau}}\mathrm{e}^{i\omega\tau}\mathrm{e}^{-a\tau}\ .\end{aligned} \tag{60}$$

After integration (see identity for Gamma function (54)), one gets

$$f^{\text{Twisted}}_{\theta_{0i},\bar\kappa}(\omega) = \left(-\frac{1}{a}\right)(\hat\omega z)^{i\omega/a}\,\Gamma\left(-\frac{i\omega}{a}\right)\mathrm{e}^{\pi\omega/2a}\left(1+\frac{\delta_{i1}\ \theta_{01}\ \omega}{az^2}\left[\frac{i\omega}{a}-1\right]\right)\ , \tag{61}$$

and

$$\omega P^{\text{Twisted}}_{\theta_{0i},\bar\kappa}(-\omega) = \frac{1}{T}\frac{1}{\mathrm{e}^{\omega/T}-1}\left(1-\frac{\delta_{i1}\ \theta_{01}\ \omega}{\pi T z^2}\right)+\mathcal{O}(\theta_{01}^2)\ , \tag{62}$$

respectively.

Of course, for deformation parameter θ_{01} approaching zero, and parameters κ and $\hat\kappa$ running to infinity, the above corrections disappear. Besides, one can observe, that for deformation parameters κ and $\hat\kappa$ going to infinity, we get the thermal spectrum for canonical Rindler space-time provided in [8].

5 Final remarks

In this article we provide linear version of three generalized twist-deformed Rindler spaces. All of them correspond to the generalized twist-deformed Minkowski space-times derived in [36]. Further, we demonstrated that there appear corrections to the Hawking thermal radiation, which are linear in deformation parameters θ^{01}, κ and $\hat\kappa$.

It should be noted, that the above results can be extended in different ways. First of all, for example, one can find the complete form of (generalized) Rindler space-times with the use of complete twist differential operators

$$\mathcal{Z}^{-1}_{\cdot,\cdot} = \exp\mathcal{O}_{\cdot,\cdot}(z,\partial_z)\ , \tag{63}$$

which appear respectively in the formulas (30), (37) and (40). However, due to the complicated form of operators $\mathcal{O}_{\cdot,\cdot}(z, \partial_z)$ such a problem seems to be quite difficult to solve from technical point of view. Besides, following the paper [11] (the case of commutative Rindler space), one can find additional physical applications for such deformed generalized Rindler space-times. The studies in these directions already started and are in progress.

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References

[1] J.D. Bekenstein, Phys. Rev. D 7, 2333 (1973)

[2] S.W. Hawking, Nature 248, 30 (1974)

[3] S.W. Hawking, Commun. Math. Phys. 43, 199 (1975)

[4] P.C.W. Davies, J. Phys. A 8, 609 (1975)

[5] W.G. Unruh, Phys. Rev. D 14, 870 (1976)

[6] W. Rindler, Am. J. Phys. 34, 1174 (1966)

[7] J. Kowalski-Glikman, Phys. Rev. D 80, 044028; arXiv: 0907.3193 [hep-th]

[8] M. Daszkiewicz, Mod. Phys. Lett. A 25, 1059 (2010)

[9] S. Majid, H. Ruegg, Phys. Lett. B 334, 348 (1994); hep-th/9405107

[10] J. Lukierski, H. Ruegg, W.J. Zakrzewski, Annals Phys. 243, 90 (1995)

[11] T. Padmanabhan, Phys. Rept. 406, 49 (2005); gr-qc/0311036

[12] P.M. Alsing, P.W. Miloni, Am. J. Phys. 72, 1524 (2004); quant-ph/041170

[13] H.S. Snyder, Phys. Rev. 72, 68 (1947)

[14] S. Coleman, S.L. Glashow, Phys. Rev. D 59, 116008 (1999)

[15] R.J. Protheore, H. Meyer, Phys. Lett. B 493, 1 (2000)

[16] F.W. Stecker, S.L. Glashow, Astroparticle Phys. 16, 97 (2001)

[17] G. Amelino-Camelia, T. Piran, Phys. Lett. B 497, 265 (2001)

[18] S. Doplicher, K. Fredenhagen, J.E. Roberts, Phys. Lett. B 331, 39 (1994); Comm. Math. Phys. 172, 187 (1995); hep-th/0303037

[19] A. Kempf and G. Mangano, Phys. Rev. D 55, 7909 (1997); hep-th/9612084

[20] A. Connes, M.R. Douglas, A. Schwarz, JHEP 9802, 003 (1998)

[21] N. Seiberg and E. Witten, JHEP 9909, 032 (1999); hep-th/9908142

[22] S. Zakrzewski, *"Poisson Structures on the Poincare group"*; q-alg/9602001

[23] Y. Brihaye, E. Kowalczyk, P. Maslanka, *"Poisson-Lie structure on Galilei group"*; math/0006167

[24] M. Chaichian, P.P. Kulish, K. Nashijima, A. Tureanu, Phys. Lett. B 604, 98 (2004)

[25] M. Daszkiewicz, Mod. Phys. Lett. A 23, 7 (2008), IFT-UWR-LV-420; arXiv: 0801.1206 [hep-th]

[26] J. Lukierski, A. Nowicki, H. Ruegg and V.N. Tolstoy, Phys. Lett. B 264, 331 (1991)

[27] J. Lukierski, A. Nowicki and H. Ruegg, Phys. Lett. B 293, 344 (1992)

[28] J. Lukierski, A. Nowicki, H. Ruegg and V.N. Tolstoy, J. Phys. A 27, 2389 (1994)

[29] J. Lukierski and M. Woronowicz, Phys. Lett. B 633, 116 (2006); hep-th/0508083

[30] S. Giller, P. Kosinski, M. Majewski, P. Maslanka and J. Kunz, Phys. Lett. B 286, 57 (1992)

[31] O. Ogievetsky, W.B. Schmidke, J. Wess, B. Zumino, Comm. Math. Phys. 150, 495 (1992)

[32] P. Aschieri, L. Castellani, A.M. Scarfone, Eur. Phys. J. C 7, 159 (1999); q-alg/9709032

[33] M. Daszkiewicz, Mod. Phys. Lett. A 23, 1757 (2008), IFT-UWR-LV-724; arXiv: 0807.0133 [hep-th]

[34] J. Lukierski and V.D. Lyakhovsky, Proceedings of the Conference *"Non-Commutative Geometry and Representation Theory in Mathematical Physics"*, Karlstad, Proceedings Serie of Contemporary Mathematics (2005); hep-th/0406155

[35] V.D. Lyakhovsky, Rep. Math. Phys. 61, 213 (2008)

[36] M. Daszkiewicz, Rep. Math. Phys. 63, 263 (2009); arXiv: 0812.1613 [mat.ph]

[37] V.G. Drinfeld, Soviet Math. Dokl. 32, 254 (1985); Algebra i Analiz (in Russian), 1, Fasc. 6, p. 114 (1989)

[38] V. Chari, A. Pressley, *"A Guide to Quantum Groups"*, Cambridge University Press, Cambridge, 1994

[39] S. Majid, *"Foundations of quantum group theory"*, Cambridge University Press, 2000

[40] L.A. Takhtajan, *"Introduction to Quantum Groups"*; in Clausthal Proceedings, Quantum groups 3-28 (see High Energy Physics Index 29 (1991) No. 12256)

[41] C. Blohmann, J. Math. Phys. 44, 4736 (2003); q-alg/0209180

[42] J. Wess, *"Deformed coordinate spaces: Derivatives"*; hep-th/0408080

[43] F.W. Byron, R.W. Fuller, *"Mathematics in Classical and Quantum Physics"*; (in Polish), PWN, 1975